

THE LOWER CENTRAL SERIES OF THE SYMPLECTIC QUOTIENT OF A FREE ASSOCIATIVE ALGEBRA

BEN BOND, DAVID JORDAN

ABSTRACT. We study the lower central series filtration L_k for a symplectic quotient $A = A_{2n}/\langle\omega\rangle$ of the free algebra A_{2n} on $2n$ generators, where $\omega = \sum[x_i, x_{i+n}]$. We construct an action of the Lie algebra H_{2n} of Hamiltonian vector fields on the associated graded components of the filtration, and use this action to give a complete description of the reduced first component $\bar{B}_1(A) = A/(L_2 + AL_3)$ and the second component $B_2 = L_2/L_3$, and we conjecture a description for the third component $B_3 = L_3/L_4$.
Keywords: Non-commutative geometry, Hamiltonian vector fields, lower central series

1. INTRODUCTION AND RESULTS

The lower central series of an associative algebra A is the descending filtration by Lie ideals, $L_1(A) := A$, and $L_k(A) := [A, L_{k-1}(A)]$. We denote by M_k the two-sided associative ideal generated by L_k , and by $B_k(A)$ and $N_k(A)$ the associated graded components $B_k(A) := L_k(A)/L_{k+1}(A)$, and $N_k(A) := M_k(A)/M_{k+1}(A)$. We also denote by \bar{B}_1 the quotient $\bar{B}_1 := A/(M_3 + L_2)$. The study of the components B_k was initiated in [FS], and continued in a series of papers [DE], [AJ], [BJ].

Let A_m denote the free algebra with generators x_1, \dots, x_m . In the algebra A_{2n} , we define:

$$\omega := \frac{1}{2} \sum_{i=1}^n [x_i, x_{i+n}],$$

and denote by $\langle\omega\rangle$ the two-sided associative ideal generated by ω .

Definition 1.1. *The symplectic quotient of the free algebra is:*

$$A'_{2n} := A_{2n}/\langle\omega\rangle.$$

In [FS], an isomorphism of associative algebras was constructed between A_m/M_3 and the algebra $\Omega_*^{even}(\mathbb{C}^m)$ of even-degree differential forms, with Fedosov product $a * b := ab + da \wedge db$. This isomorphism

maps $\omega \in A_{2n}$ to the standard symplectic form on \mathbb{C}^{2n} which, by abuse of notation, we also denote ω .

The study of the components $B_k(A_m)$ and $N_k(A_m)$ has relied heavily upon an action of the Lie algebra, W_m , of polynomial vector fields on \mathbb{C}^m . It is shown in [DE] that each B_k has a finite-length Jordan-Hölder series with respect to this action, whose composition factors are so-called “tensor field modules”. In [AJ], and [BJ], bounds are given on the degree of these modules, which allow the components to be computed explicitly in many examples.

In the present paper, we construct an action of the Lie algebra H_{2n} of Hamiltonian vector fields (i.e. vector fields which fix the form ω) on the components $B_k(A'_{2n})$ and $N_k(A'_{2n})$. By studying this action, we are able to generalize many of the results about A_m to the symplectic quotients A'_{2n} .

In particular, the general framework discussed in Section 3 yields isomorphisms,

$$\begin{aligned} A'_{2n}/M_3(A'_{2n}) &\cong \Omega_*^{even}(\mathbb{C}^{2n})/\langle\omega\rangle, \\ \bar{B}_1(A'_{2n}) &\cong \Omega_*^{even}(\mathbb{C}^{2n})/(\Omega_{closed}^{even,+}(\mathbb{C}^{2n}) + \langle\omega\rangle), \\ B_2(A'_{2n}) &\cong \Omega_{closed}^{even,+}(\mathbb{C}^{2n})/(\Omega_{closed}^{even,+}(\mathbb{C}^{2n}) \cap \langle\omega\rangle). \end{aligned}$$

The irreducible representations of H_{2n} which appear in the present work are certain tensor field modules \mathcal{F}_λ associated to Young diagrams $\lambda \neq (1^k)$, and the irreducible sub-quotients of $\mathcal{F}_{(1^k)}$ (see Section 2.4 for details). In fact, we show

Proposition 1.2. *As H_{2n} -modules, each $B_k(A'_{2n})$ and $N_k(A'_{2n})$, for $k \geq 2$, has a finite length Jordan Hölder series, consisting of tensor field modules \mathcal{F}_λ , with $\lambda \neq (1^k)$, and of subquotients $\mathcal{F}_{(1^k)}/T_k$, Y_k/X_k , Z_k/X_k , X_k of $\mathcal{F}_{(1^k)}$.*

Our main results are a computation of the Jordan-Hölder series for the modules A'_{2n}/M_3 , $\bar{B}_1(A'_{2n})$, $B_2(A'_{2n})$, and conjecturally for $B_3(A'_{2n})$. We have:

Theorem 1.3. *The H_{2n} -module composition factors of $A'_{2n}/M_3(A'_{2n})$ are:*

$$\begin{aligned} \mathcal{F}_{(1^k)}/T_k, Y_k/X_k, Z_k/X_k, X_k & \quad \text{for } k \text{ even, } 2 \leq k \leq n-1, \\ \mathcal{F}_{(1^n)}/X_n, X_n & \quad \text{if } n \text{ even.} \\ \mathcal{F}_0/X_0, X_0. & \end{aligned}$$

Theorem 1.4. *The H_{2n} -module composition factors of $\bar{B}_1(A'_{2n})$ are:*

$$\begin{aligned} \mathcal{F}_{(1^k)}/T_k, Y_k/X_k & \quad \text{for } k \text{ even, } 2 \leq k \leq n-1, \\ \mathcal{F}_{(1^n)}/X_n & \quad \text{if } n \text{ even.} \\ \mathcal{F}_0/X_0, X_0. & \end{aligned}$$

Theorem 1.5. *The H_{2n} -module composition factors of $B_2(A'_{2n})$ are:*

$$\begin{aligned} Z_k/X_k, X_k & \quad \text{for } k \text{ even, } 2 \leq k \leq n-1, \\ X_n & \quad \text{if } n \text{ even.} \end{aligned}$$

Conjecture 1.6. *The H_{2n} -module composition factors of $B_3(A'_{2n})$ are:*

$$\mathcal{F}_{(2,1^k)}, \mathcal{F}_{(1^k)}/T_k, Z_k/X_k \quad \text{for } k \text{ odd, } 1 \leq k \leq n-1.$$

Remark 1.7. In Section 5, we show that the above list is an upper bound for the Jordan-Hölder series of $B_3(A'_{2n})$, so that the only ambiguity is whether the summands do in fact appear. For $2n = 4, 6$, the conjecture is a theorem, based on MAGMA computations showing that certain cyclic generators of each term in the series are non-zero.

The outline of this paper is as follows: In Section 2, we recall facts from the representation theory of the Lie algebras W_n and H_{2n} which we will need. In Section 3, we construct an action of the Lie algebra H_{2n} on each quotient $B_k(A'_{2n})$, and $\bar{B}_1(A'_{2n})$, and show that these are finite extensions of tensor field modules. In Section 4, we prove Theorems 1.3, 1.4, and 1.5. In Section 5 we describe $B_3(A'_{2n})$ as a H_{2n} -module, and conjecture a description of $B_3(A'_{2n})$.

1.1. Acknowledgments. The authors would like to thank Pavel Etingof and Xiaoguang Ma for many helpful conversations during the course of this project; in particular the contents of Section 2.3 were explained to us by Pavel Etingof, who also conjectured Proposition 3.4. Crucial evidence was collected using the Magma computational algebra system [BCP]. Finally, we are grateful to a careful referee for providing many helpful comments, and corrections to formulas in Proposition 2.10 and 5.5.

2. PRELIMINARIES

In this section we recall the Lie algebras W_m , H_{2n} , and \mathfrak{sp}_{2n} of polynomial vector fields on \mathbb{C}^m , Hamiltonian vector fields on \mathbb{C}^{2n} , and Hamiltonian linear transformations of \mathbb{C}^{2n} , respectively.

2.1. The symplectic Lie algebra and the restriction functor.

Let $E_{ij} \in \mathfrak{gl}_{2n}$ be the matrix with 1 in the i -th row and j -th column and 0 everywhere else. Let L_i be the dual basis to the diagonal span $\{E_{ii}\}_i$. The Cartan subalgebra of \mathfrak{sp}_{2n} is generated by:

$$H_i := E_{i,i} - E_{i+n,i+n}, \text{ for } 1 \leq i \leq n.$$

The positive roots are

$$\{L_i - L_j\}_{1 \leq i < j \leq n} \cup \{L_i + L_j\}_{1 \leq i \leq j \leq n}.$$

The positive root vectors, with corresponding roots, are:

$$\begin{aligned} X_{ij} &:= E_{ij} - E_{j+n,i+n} & L_i - L_j & \text{ for } i < j \\ Y_{ij} &:= E_{i,j+n} + E_{j,i+n} & L_i + L_j & \text{ for } i < j \\ U_i &:= E_{i,i+n} & 2L_i. \end{aligned}$$

To simplify notation later in the paper, we abbreviate $X_{ii} := H_i$, $Y_{ii} := U_i$. For $i = 1, \dots, n$, we denote the i th fundamental weight $\rho_i(H_j) := 1$ if $j \leq i$, 0 else.

Recall that the irreducible representations of \mathfrak{sp}_{2n} are parameterized by Young diagrams with at most n rows. To reduce notational clutter, we will use the same notation, $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$, to refer both to the Young diagram, and the corresponding irreducible representation of \mathfrak{sp}_{2n} . To avoid confusion, we denote by $\underline{\mu} = [\mu_1 \geq \dots \geq \mu_m]$ the corresponding irreducible representation of \mathfrak{gl}_m . We denote by $Y(k)$ the set of Young diagrams with at most k rows.

Let us recall the restriction functor,

$$\text{Res}_{\mathfrak{sp}_{2n}}^{\mathfrak{gl}_{2n}} : \mathfrak{gl}_{2n}\text{-mod} \rightarrow \mathfrak{sp}_{2n}\text{-mod}.$$

The restriction formula describes the restriction of simple \mathfrak{gl}_{2n} modules; it is as follows. For $\lambda \in Y(n)$, and $\mu \in Y(2n)$, we define:

$$N_{\lambda\mu} = \sum_{\eta} N_{\eta\lambda\mu},$$

where $N_{\eta\lambda\mu}$ is the Littlewood-Richardson coefficient [FH], and the sum ranges over partitions $\eta = (\eta_1 = \eta_2 \geq \eta_3 = \eta_4 \geq \dots) \in Y(2n)$ in which each part appears an even number of times. Then we have:

Theorem 2.1 ([FH], p. 427). *The restriction from \mathfrak{gl}_{2n} to \mathfrak{sp}_{2n} of the representation $\underline{\mu}$ is:*

$$\text{Res}_{\mathfrak{sp}_{2n}}^{\mathfrak{gl}_{2n}}(\underline{\mu}) = \bigoplus_{\lambda \in Y(n)} N_{\lambda\mu} \lambda.$$

2.2. The Lie algebras W_{2n} and H_{2n} .

Definition 2.2. *The Lie algebra of polynomial vector fields on \mathbb{C}^m is $W_m := \text{Der } \mathbb{C}[x_1, \dots, x_m]$.*

Definition 2.3. *The Lie algebra of Hamiltonian vector fields on \mathbb{C}^{2n} is the Lie subalgebra:*

$$H_{2n} = \{D \in W_{2n} \mid D\omega = 0\},$$

of polynomial vector fields that preserve the symplectic form ω .

Any $w \in W_m$ may be written $w = \sum_i f_i \frac{\partial}{\partial x_i}$, with $f_i \in \mathbb{C}[x_1, \dots, x_m]$. Likewise, an arbitrary element of H_{2n} can be written in the form:

$$D_u := \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_{i+n}} - \frac{\partial u}{\partial x_{i+n}} \frac{\partial}{\partial x_i} \right)$$

The commutator relation is $[D_u, D_v] = D_{\{u, v\}}$, where $\{u, v\}$ is the Poisson bracket,

$$\{u, v\} = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_{i+n}} - \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_{i+n}} \right).$$

We define a grading on W_m by letting W_m^k be the span of vector fields with $\deg f_i = k + 1$ for each $i = 1, \dots, m$. This grading is inherited by H_{2n} ; it is easy to check that H_{2n}^k is spanned by elements D_u with $\deg u = k + 2$. We let $W_m^{\geq k} := \bigoplus_{j \geq k} W_m^j$ and $H_{2n}^{\geq k} := \bigoplus_{j \geq k} H_{2n}^j$ denote the trivially induced filtrations. We may identify W_m^0 and H_{2n}^0 with \mathfrak{gl}_m and \mathfrak{sp}_{2n} , respectively.

2.3. Tensor field modules. We have projections $H_{2n}^{\geq 0} \rightarrow H_{2n}^0 \cong \mathfrak{sp}_{2n}$, and $W_m^{\geq 0} \rightarrow W_m^0 \cong \mathfrak{gl}_m$, through which we can pull back representations. In this way we can define the functors of co-induction:

$$\text{CoInd}_{\mathfrak{sp}_{2n}}^{H_{2n}} : \mathfrak{sp}_{2n}\text{-mod} \rightarrow H_{2n}\text{-mod},$$

$$V \mapsto \text{Hom}_{H_{2n}^{\geq 0}}^{fin}(U(H_{2n}), V),$$

$$\text{CoInd}_{\mathfrak{gl}_m}^{W_m} : \mathfrak{gl}_m\text{-mod} \rightarrow W_m\text{-mod},$$

$$V \mapsto \text{Hom}_{W_m^{\geq 0}}^{fin}(U(W_m), V),$$

where Hom^{fin} denotes the homomorphisms with finite-dimensional support, and U is the universal enveloping algebra.

Definition 2.4. For λ (resp. $\underline{\mu}$) an irreducible representations of \mathfrak{sp}_{2n} (resp. \mathfrak{gl}_m), we define:

$$\mathcal{F}_\lambda := \text{CoInd}_{\mathfrak{sp}_{2n}}^{H_{2n}} \lambda,$$

$$\mathcal{G}_{\underline{\mu}} := \text{CoInd}_{\mathfrak{gl}_{2n}}^{W_{2n}} \underline{\mu}.$$

We have $H_{2n}^{-1} = \mathbb{C}\partial_1 \oplus \cdots \oplus \mathbb{C}\partial_{2n}$, and $W_m^{-1} = \mathbb{C}\partial_1 \oplus \cdots \oplus \mathbb{C}\partial_m$. We thus obtain isomorphisms,

$$\mathcal{F}_\lambda \cong (\mathbb{C}[[x_1, \dots, x_{2n}]] \otimes \lambda)^{fin} \cong \mathbb{C}[x_1, \dots, x_{2n}] \otimes \lambda$$

$$\mathcal{G}_{\underline{\mu}} \cong (\mathbb{C}[[x_1, \dots, x_m]] \otimes \underline{\mu})^{fin} \cong \mathbb{C}[x_1, \dots, x_m] \otimes \underline{\mu}$$

Theorem 2.5 ([R], p. 478). *If $\lambda \neq (1^k)$, then \mathcal{F}_λ is irreducible.*

Proof. In [R], it is proven that the induced modules,

$$\text{Ind}_{\mathfrak{sp}_{2n}}^{H_{2n}} V_\lambda := U(H_{2n}) \otimes_{H_{2n}^{\geq 0}} V_\lambda,$$

are irreducible for $\lambda \neq (1^k)$. We have the duality pairing between induction and co-induction:

$$\text{Hom}_{H_{2n}^{\geq 0}}^{fin}(U(H_{2n}), V^*) \otimes (U(H_{2n}) \otimes_{H_{2n}^{\geq 0}} V) \rightarrow \mathbb{C}$$

$$f \otimes (a \otimes b) \mapsto \langle f(a), b \rangle.$$

This implies that $\mathcal{F}_\lambda \cong \text{CoInd}_{\mathfrak{sp}_{2n}}^{H_{2n}} V_\lambda^*$ is irreducible. \square

The modules $\mathcal{F}_{(1^k)}$ are not irreducible; to describe their structure, we begin by realizing them as submodules in $\Omega^k(\mathbb{C}^{2n})$, as follows. Recall from symplectic Hodge theory (see [G] for details) that $\Omega^\bullet(\mathbb{C}^{2n})$ carries an action of the Lie algebra \mathfrak{sl}_2 , where E acts by contraction, ι_π , with the generating Poisson bi-vector, F acts by wedging with ω , and H acts diagonally: $H\eta = (n - k)\eta$, for $\eta \in \Omega^k(\mathbb{C}^{2n})$.

The \mathfrak{sl}_2 -action clearly commutes with the H_{2n} -action, and $\mathcal{F}_{(1^k)}$ is the space of k -forms lying in the kernel of ι_π , i.e. the subspace of \mathfrak{sl}_2 -singular vectors of weight $n - k$. In addition to the differential d , we have the operator δ defined by the equation, $\delta = *d*$, where $*$ is the symplectic Hodge star operator.

We define submodules X_k, Y_k, Z_k, T_k of $\mathcal{F}_{(1^k)}$ as follows. The operators d and δ generate a copy of $V_{(1)}$ under the \mathfrak{sl}_2 -action, where $V_{(i)}$ is the $i + 1$ dimensional irreducible representation of \mathfrak{sl}_2 . Thus, upon restricting the map $d : \Omega^k \rightarrow \Omega^{k+1}$ to the $V_{(n-k)}$ -isotypic component, we find that its image lies in a submodule isomorphic to $(V_{(1)} \otimes V_{(n-k)})$. Recalling the isomorphism $V_{(1)} \otimes V_{(n-k)} \cong V_{(n-k-1)} \oplus V_{(n-k+1)}$ (for $k < n$),

we define:

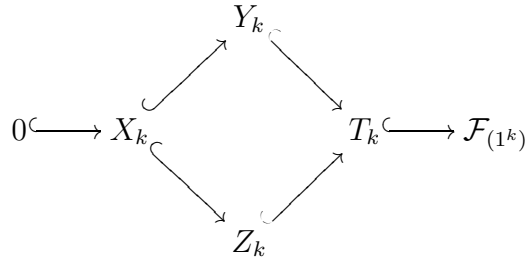
$$\begin{aligned} X_k &= \{x \in \mathcal{F}_{(1^k)} \mid dx = 0\}, \\ Y_k &= \{x \in \mathcal{F}_{(1^k)} \mid dx \text{ generates } V_{(n-k-1)} \text{ or } 0 \text{ under the } \mathfrak{sl}_2\text{-action}\}, \\ Z_k &= \{x \in \mathcal{F}_{(1^k)} \mid dx \text{ generates } V_{(n-k+1)} \text{ or } 0 \text{ under the } \mathfrak{sl}_2\text{-action}\}, \\ T_k &= Y_k + Z_k. \end{aligned}$$

Remark 2.6. Note that T_k is a proper submodule of \mathcal{F}_k because

$$T_k \cap \ker d \neq Y_k \cap \ker d + Z_k \cap \ker d.$$

As the operators d, δ commute with H_{2n} , the subspaces X_k, Y_k, Z_k, T_k are H_{2n} -submodules.

Theorem 2.7. *For $1 \leq k \leq n-1$, the complete lattice of H_{2n} -submodules for $\mathcal{F}_{(1^k)}$ is as follows:*



In particular, the Jordan-Hölder series of $\mathcal{F}_{(1^k)}$ is $X_k, Y_k/X_k, Z_k/X_k, \mathcal{F}_{(1^k)}/T_k$.

Proof. It is straightforward to check that each containment in the theorem is proper. In [R], it is proven that the induced modules, $\mathcal{F}_{(1^k)}^*$, have length four; thus so do the $\mathcal{F}_{(1^k)}$, and the theorem follows. \square

Theorem 2.8. *For $k = 0, n$, $\mathcal{F}_{(1^k)}$ has the closed forms X_k as a submodule; the quotient is irreducible.*

Proof. For $k = 0$, it well-known that functions modulo constants form an irreducible H_{2n} -module. For $k = n$, it is shown in [R] that $\mathcal{F}_{(1^n)}$ has length two, as an H_{2n} -module; the closed forms clearly form a proper submodule. \square

It follows from Definition 2.4 that restriction and coinduction commute:

$$\text{Res}_{H_{2n}}^{W_{2n}} \text{CoInd}_{\mathfrak{gl}_{2n}}^{W_{2n}} V \cong \text{CoInd}_{\mathfrak{sp}_{2n}}^{H_{2n}} \text{Res}_{\mathfrak{sp}_{2n}}^{\mathfrak{gl}_{2n}} V.$$

In particular, we have from Theorem 2.1 the restriction formula:

$$(1) \quad \text{Res}_{H_{2n}}^{W_{2n}}(\mathcal{G}_{\underline{\mu}}) \cong \bigoplus_{\lambda} N_{\lambda\mu} \mathcal{F}_{\lambda}.$$

2.4. Distinguished cyclic vectors. Recall that the tensor field modules \mathcal{F}_λ are irreducible when $\lambda \neq (1^k)$, and that we have an isomorphism of \mathfrak{sp}_{2n} -modules,

$$\mathcal{F}_\lambda \cong (\mathbb{C}[[x_1, \dots, x_{2n}]] \otimes \lambda)^{fin} \cong \mathbb{C}[x_1, \dots, x_{2n}] \otimes \lambda.$$

In particular, we observe that \mathcal{F}_λ contains a unique-up-to-scalars \mathfrak{sp}_{2n} -highest weight vector v_λ of weight λ . The following proposition follows immediately:

Proposition 2.9. *Let $\lambda \neq (1^k) \in Y(n)$. A tensor field module M for H_{2n} is isomorphic to \mathcal{F}_λ if, and only if, M contains a highest weight vector v_λ for \mathfrak{sp}_{2n} , such that $\partial_i v_\lambda = 0$ for all $i = 1, \dots, 2n$.*

We call any such highest weight vector $v_\lambda \in \mathcal{F}_\lambda$ a distinguished cyclic vector. Note that, in [R], the term “singular vector” is used in the dual context.

The situation for $\mathcal{F}_{(1^k)}$ is more complicated.

Proposition 2.10. *We have:*

- (1) *The H_{2n} -submodule $X_k \subset \mathcal{F}_{(1^k)} \subset \Omega^k(\mathbb{C}^{2n})$ is generated by:*

$$\underline{x}_k = dx_1 \cdots dx_k.$$

- (2) *The H_{2n} -submodule $Y_k \subset \mathcal{F}_{(1^k)} \subset \Omega^k(\mathbb{C}^{2n})$ is generated by:*

$$\underline{y}_k = \sum_{i=1}^{k+1} (-1)^i x_i dx_1 \cdots \widehat{dx_i} \cdots dx_{k+1}.$$

- (3) *The H_{2n} -submodule $Z_k \subset \mathcal{F}_{(1^k)} \subset \Omega^k(\mathbb{C}^{2n})$ is generated by:*

$$\underline{z}_k = \omega \wedge \underline{y}_{k-2} - 2(n - k + 2)y \wedge \underline{x}_{k-1}.$$

where $y = \frac{1}{2} \sum_i (x_i dx_{i+n} - x_{i+n} dx_i)$ is the Liouville form of ω .

Proof. It is well-known that each of X_k, Y_k, Z_k is generated as H_{2n} by the unique \mathfrak{sp}_{2n} -highest weight vector of weight $(1^k), (1^{k+1}), (1^{k-1})$ and degree $k, k+1, k+1$, respectively. It is straightforward to check that each of $\underline{x}_k, \underline{y}_k$, and \underline{z}_k is highest weight of the correct weight and degree, and that the $\underline{x}_k, \underline{y}_k$ are in X_k, Y_k , respectively. It remains only to show that $\underline{z}_k \in Z_k$, which follows from the identities:

$$\begin{aligned} i_\pi(\omega \underline{y}_{k-2}) &= (n - k + 2) \underline{y}_{k-2}, & i_\pi(y \underline{x}_{k-1}) &= \frac{\underline{y}_{k-2}}{2}, \\ i_\pi(d \underline{z}_k) &= (n - k + 1)(k - 3 - 2n) \underline{x}_{k-1} \neq 0. \end{aligned}$$

□

Proposition 2.11. *We have the following identities:*

$$\begin{aligned} \partial_l \underline{y}_k &= \begin{cases} (-1)^k X_{l,k+1}^T \underline{x}_k, & 1 \leq l \leq k, \\ (-1)^{k+1} \underline{x}_k, & l = k+1, \\ 0, & k+2 \leq l \leq 2n. \end{cases} \\ \partial_l \underline{z}_k &= \begin{cases} (-1)^k \sum_{j \geq k} Y_{lj}^T X_{kj}^T \underline{x}_k, & 1 \leq l \leq k-1, \\ (-1)^k (n-k+2) Y_{k,l}^T \underline{x}_k, & k \leq l \leq n, \\ 0, & 1 \leq l-n \leq k-1, \\ (-1)^{k-1} (n-k+2) X_{k,l-n}^T \underline{x}_k, & k \leq l-n \leq n. \end{cases} \end{aligned}$$

Proof. We compute:

$$\begin{aligned} \partial_l \underline{y}_k &= \begin{cases} (-1)^l dx_1 \cdots \widehat{dx_l} \cdots dx_{k+1}, & 1 \leq l \leq k, \\ (-1)^{k+1} dx_1 \cdots dx_k, & l = k+1, \\ 0, & k+2 \leq l \leq 2n. \end{cases} \\ \partial_l \underline{z}_k &= \begin{cases} (-1)^l \omega dx_1 \cdots \widehat{dx_l} \cdots dx_{k-1} - (n-k+2) dx_{n+l} dx_1 \cdots dx_{k-1}, & 1 \leq l \leq k-1, \\ -(n-k+2) dx_{l+n} dx_1 \cdots dx_{k-1}, & k \leq l \leq n, \\ 0, & 1 \leq l-n \leq k-1, \\ dx_1 \cdots dx_{k-1} dx_{l-n}, & k \leq l-n \leq n. \end{cases} \end{aligned}$$

The claims now follow by direct comparison. \square

3. ACTION OF H_{2n} ON $B_k(A'_{2n})$

In this section, we construct an action of H_{2n} on each $B_k(A'_{2n})$, echoing that of W_{2n} on each $B_k(A_{2n})$.

Lemma 3.1. *Let I be an ideal of an associative algebra A , and let $\text{Im}(L_k(A) \cap I)$ denote the image of $L_k(A) \cap I$ in $B_k(A)$ then*

$$B_k(A/I) = B_k(A) / \text{Im}(L_k(A) \cap I).$$

Proof. Lemma 2.4 of [BB] states the result in the case $k = 2$, while the same proof applies to all k . \square

Proposition 3.2. *The W_{2n} action on each $B_k(A_{2n})$ and $N_k(A_{2n})$ descends to an action of H_{2n} on $B_k(A'_{2n})$ and $N_k(A'_{2n})$.*

Proof. From Lemma 3.1, $B_k(A'_{2n}) = B_k(A_{2n}) / (L_k(A_{2n}) \cap \langle \omega \rangle)$. Since both $L_k(A_{2n})$ and $\langle \omega \rangle$ are invariant under action by H_{2n} , so is their intersection; thus, the action descends. \square

Corollary 3.3. *We have the following isomorphisms:*

- (1) $B_2(A'_{2n}) \cong \Omega_{\text{closed}}^{\text{even},+}(\mathbb{C}^{2n}) / \langle \omega \rangle$.
- (2) $A'_{2n} / M_3(A'_{2n}) \cong \Omega_*^{\text{even}}(\mathbb{C}^{2n}) / \langle \omega \rangle$.
- (3) $\bar{B}_1(A'_{2n}) \cong (\Omega^{\text{even}}(\mathbb{C}^{2n}) / \Omega_{\text{closed}}^{\text{even}}(\mathbb{C}^{2n})) / \langle \omega \rangle$.

Proof. Recall that under the Fedosov product, $\omega \in A_{2n}$ maps to the standard symplectic form on \mathbb{C}^{2n} , which we also denote as ω . For (1), let $A = A_{2n}$, $I = \langle \omega \rangle$ in Lemma 3.1, which gives $B'_{2,2n} \cong B_{2,2n}/(L_2 \cap \langle \omega \rangle)$. The result follows from the isomorphism given in [FS], $B_{2,2n} \cong \Omega_{closed}^{even,+}(\mathbb{C}^{2n})$. The proof of (3) follows similarly. For (2), $A_{2n}/M_3 \cong \Omega_*^{even}(\mathbb{C}^{2n})$ as shown in [FS]. The result follows by application of the third isomorphism theorem. \square

3.1. Finite length Jordan-Hölder series.

Proposition 3.4. *The H_{2n} -module Jordan-Hölder series of $B_k(A'_{2n})$ is finite length, and is composed of \mathcal{F}_λ , and of the irreducible submodules of the reducible $\mathcal{F}_{(1^k)}$.*

Proof. The main result of [DE] asserts that the Jordan-Hölder series of $B_k(A_{2n})$, for $k \geq 3$ as a W_{2n} -module consists of tensor field modules \mathcal{G}_μ for a finite set of $\underline{\mu} \in Y_{2n}$. Thus the Jordan-Hölder series of $\text{Res}_{H_{2n}}^{W_{2n}} B_k(A_{2n})$ consists of a finite set of \mathcal{F}_λ , according to the restriction rules (1), which are finite-to-one. For all λ except $\lambda = (1^k)$, \mathcal{F}_λ is irreducible, while each $\mathcal{F}_{(1^k)}$ has finite length Jordan-Hölder series by Theorems 2.7, 2.8; thus $B_k(A_{2n})$ is of finite length. Thus when we quotient by $\langle \omega \rangle$, we find that $B_k(A'_{2n})$ also has finite length. \square

4. STRUCTURE OF A'_{2n}/M_3 , $\bar{B}_1(A'_{2n})$, AND $B_2(A'_{2n})$

Recall from Equation (1) that the space of k -forms, $\Omega^k(\mathbb{C}^{2n}) \cong \mathcal{G}_{(1^k)}$ decomposes as an H_{2n} -module:

$$\Omega^k(\mathbb{C}^{2n}) \cong \bigoplus_{s \leq \frac{k}{2}} \mathcal{F}_{(1^{k-2s})},$$

and that each $\mathcal{F}_{(1^{k-2s})}$, for $s \geq 1$, consists of forms divisible by ω . On the other hand, $\mathcal{F}_{(1^k)}$ consists of locally finite \mathfrak{sl}_2 -singular vectors of weight $n - k$, and so cannot lie in the image of $F = \omega \wedge -$. Thus we have $\Omega^k(\mathbb{C}^{2n})/\langle \omega \rangle \cong \mathcal{F}_{(1^k)}$.

Proposition 4.1. *For $\leq k \leq n - 1$, we have $Z_k = \mathcal{F}_{(1^k)} \cap \langle \omega \rangle + X_k$.*

Proof. First, we give an alternate description of Z_k . Consider the H_{2n} -submodule $\tilde{Z}_k \subset \mathcal{F}_{(1^k)}$, consisting of those α satisfying $d\alpha \in \Omega_{ex}^{k-1} \cdot \langle \omega \rangle$. Clearly $\tilde{Z}_k \supsetneq X_k$, and $\tilde{Z}_k \cap Y_k = X_k$. According to Theorem 2.7, Z_k is the unique such H_{2n} -submodule, and so $\tilde{Z}_k = Z_k$.

The containment \supseteq is clear, using the alternate description of Z_k . Conversely, suppose that $d\alpha = (d\nu)\omega$. Then, integration by parts gives:

$$\alpha = \nu\omega - \nu d\omega + d\eta = \nu\omega + d\eta,$$

for some exact form $d\eta$. \square

Corollary 4.2. *We have the following:*

$$\begin{aligned} A'_{2n}/M_3 &\cong \bigoplus_{\substack{k \text{ even} \\ 0 \leq k \leq n}} \mathcal{F}_{(1^k)} \\ \bar{B}_1(A'_{2n}) &\cong \mathcal{F}_{(0)} \oplus \bigoplus_{\substack{k \text{ even} \\ 2 \leq k \leq n}} \mathcal{F}_{(1^k)}/Z_k \\ B_2(A'_{2n}) &\cong \bigoplus_{\substack{k \text{ even} \\ 2 \leq k \leq n}} Z_k. \end{aligned}$$

Theorems 1.3, 1.4, and 1.5 follow from the corollary, and Theorems 2.7 and 2.8.

5. STRUCTURE OF $B_3(A'_{2n})$

In this section, we give a complete description of $B_3(A_{2n})$ as an H_{2n} -module, and use this to conjecture a description of $B_3(A'_{2n})$. Theorem 1.8 of [AJ] gives the following decomposition:

$$B_3(A_{2n}) \cong \bigoplus_{i=1}^n \mathcal{G}_{\underline{(2, 1^{2i-1})}}.$$

By Theorem 2.1, for $k \leq n-1$, odd, we have:

$$\text{Res}_{H_{2n}}^{W_{2n}} \mathcal{G}_{\underline{(2, 1^k)}} = \bigoplus_{s=0}^{(k+1)/2} (\mathcal{F}_{(2, 1^{k-2s})} \oplus \mathcal{F}_{(1^{k-2s})})$$

Remark 5.1. Recall that we have a surjection $A/M_3 \otimes B_2 \rightarrow B_3$ given by $a \otimes b \mapsto [a, b]$, relying on the containment $[M_3, L_2] \subset L_4$, proved in [FS]. We have the Feigin-Shoikhet isomorphisms $A_{2n}/M_3 \cong \Omega^{ev}(\mathbb{C}^{2n})$, $B_2(A_{2n}) \cong \Omega_{ex}^{+, ev}(\mathbb{C}^{2n})$. Thus we have a surjection,

$$\Omega^{ev}(\mathbb{C}^{2n}) \otimes \Omega_{ex}^{ev}(\mathbb{C}^{2n}) \rightarrow B_3(A_{2n}).$$

We abuse notation and write $[a, b]$ for the image of $a \otimes b$.

Lemma 5.2. *For k odd, and $0 \leq s \leq (k-1)/2$, the submodule*

$$\mathcal{F}_{(2, 1^{k-2s})} \subset \text{Res}_{H_{2n}}^{W_{2n}} \mathcal{G}_{\underline{(2, 1^k)}} \subset B_3(A_{2n}),$$

is generated by $v_{k,s} = [x_1, \xi]$, where $\xi = dx_1 \wedge \dots \wedge dx_{k-2s+1} \omega^s$

Proof. Notice ξ is an even, closed form, so that we have $v_{k,s} \in L_3$. The vectors x_1 and ξ , and thus $x_1 \otimes \xi$, are clearly highest weight vectors for \mathfrak{sp}_{2n} . We have $\partial_i v_{k,s} = 0$, for any $i = 1, \dots, 2n$.

It only remains to show that $v_{k,s} \notin L_4$. Notice that

$$2^{-(k-2s+1)/2} [x_1, x_2 [x_3, x_4 [\cdots [x_{k-2s}, x_{k-2s+1} \omega^s] \cdots]]$$

maps to ξ under the Feigin-Shoikhet isomorphism. The proof is very similar to that of Proposition 5.11 of [AJ]: we find an algebra B , and a map $\theta : A_{2n} \rightarrow B$ in which we can compute directly that $\theta(v_{k,s}) \notin L_4(B)$. We let $B = A \otimes E$, where A is the free algebra on two generators a, b , E is the exterior algebra with generators z_0, \dots, z_{2n} . We define θ by $\theta(x_1) = ez_0 + fz_1$, and $\theta(x_i) = z_i$ for $i \geq 2$.

We compute:

$$\begin{aligned} \theta(\omega) &= ez_0 z_{1+n} + fz_1 z_{1+n} + \sum_{i \geq 2} z_i z_{i+n}, \\ \theta(\omega^s) &= s(ez_0 z_{1+n} + fz_1 z_{1+n}) \left(\sum_{i \geq 2} z_i z_{i+n} \right)^{s-1} + \left(\sum_{i \geq 2} z_i z_{i+n} \right)^s \\ \theta(v_{k,s}) &= 4[e, f]z_0 \cdots z_{k-2s+1} \left(\sum_{i \geq 2} z_i z_{i+n} \right)^s. \end{aligned}$$

By applying Corollary 5.10 of [AJ], we see that $v_{k,s}$ is nonzero in $B_3(A_{2n})$, and thus is a distinguished cyclic generator of $\mathcal{F}_{(2,1^{k-2s})}$. \square

Corollary 5.3. *All summands $\mathcal{F}_{(2,1^{k-2s})} \subset \text{Res}_{H_{2n}}^{W_{2n}} \mathcal{G}_{(2,1^k)}$, except $\mathcal{F}_{(2,1^k)}$, are zero in $B_3(A'_{2n})$.*

Proof. Clearly $v_{k,s} \in \langle \omega \rangle$ if, and only if, $s > 0$. \square

To find the singular vectors corresponding to summands $\mathcal{F}_{(1^l)}$, inside $B_3(A_{2n})$, we introduce the following homomorphisms of \mathfrak{sp}_{2n} -modules:

$$\begin{aligned} \phi_s &: \Omega^{ev} \otimes \Omega_{ex}^{odd} \rightarrow B_3(A_{2n}), \\ u \otimes v &\mapsto \sum_i [x_i u, v dx_{i+n} \omega^s] - [x_{i+n} u, v dx_i \omega^s], \\ \psi_s &: \Omega^{ev} \rightarrow B_3(A_{2n}), \\ v &\mapsto \sum_i [\omega^s x_{i+n}, d(v x_i)] - [\omega^s x_i, d(v x_{i+n})]. \end{aligned}$$

We note that constant vector fields do not commute with ϕ_s, ψ_s ; rather, we have:

$$\begin{aligned} \partial_l \phi_s(u \otimes v) &= \begin{cases} \phi_s(\partial_l(u \otimes v)) + [u, v dx_{l+n} \omega^s], & 1 \leq l \leq n \\ \phi_s(\partial_l(u \otimes v)) - [u, v dx_{l-n} \omega^s], & n+1 \leq l \leq 2n \end{cases} \\ \partial_l \psi_s(v) &= \begin{cases} \psi_s(\partial_l v) + [\omega^s x_{l+n}, dv], & 1 \leq l \leq n \\ \psi_s(\partial_l v) - [\omega^s x_{l-n}, dv], & n+1 \leq l \leq 2n \end{cases}. \end{aligned}$$

We also define the following elements of $\Omega(\mathbb{C}^{2n})$:

$$\begin{aligned} a_k &= dx_1 \wedge \dots \wedge dx_k \\ p_{j,m} &= (-1)^j dx_1 \wedge \dots \wedge \hat{dx}_j \wedge \dots \wedge dx_m, \text{ or } 0 \text{ when } m < j. \\ q_m &= \sum_{j=1}^m x_j p_{j,m} \end{aligned}$$

We collect here several easily proven observations for later use:

Proposition 5.4. *The vector q_m is a \mathfrak{sp}_{2n} -highest weight vector of weight ρ_m , and we have the following identities:*

$$p_{m,m} = (-1)^m a_{m-1}, \quad dq_m = ma_m, \quad \partial_l q_m = \begin{cases} p_{l,m}, & \text{if } l \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

We now construct distinguished cyclic vectors $\bar{x}_{k,s}$, $\bar{y}_{k,s}$, $\bar{z}_{k,s}$ for the summands $\mathcal{F}_{(1^{k-2s})} \subset (2, 1^k) \subset B_3(A_{2n})$.

Theorem 5.5. *Let $1 \leq k \leq n$ be an odd integer. The distinguished cyclic vectors for the H_{2n} -submodule,*

$$\mathcal{F}_{(1^{k-2s})} \subset \text{Res}_{H_{2n}}^{W_{2n}} \mathcal{G}_{(2,1^k)} \subset B_3(A_{2n}),$$

are given as follows:

$$\begin{aligned} \bar{x}_{k,s} &= \phi_s(1 \otimes a_{k-2s}) \\ \bar{y}_{k,s} &= \begin{cases} \frac{k-2s+1}{k-4s} \phi_s \left(\sum_{j=1}^{k-2s+1} x_j \otimes p_{j,k-2s+1} \right) - \frac{\psi_s(q_{k-2s+1})}{k-4s} & \text{if } k-2s \neq n-1 \\ \sum_{j=1}^n [\tilde{p}_{j,n}, x_j \omega^{s+1}] & \text{if } k-2s = n-1 \end{cases} \\ \bar{z}_{k,s} &= \begin{cases} \sum_{i=1}^n [x_i, d(x_{i+n}y) \omega^s] - [x_{i+n}, d(x_i y) \omega^s], & \text{if } k-2s = 1 \\ \frac{2n-3k+6s+5}{k-2s-1} y_{k,s+1} - \frac{2(n-k+2s+2)}{k-2s-1} \phi_s(1 \otimes d(q_{k-2s-1}y)) & \text{if } k-2s \neq 1 \end{cases} \end{aligned}$$

The proof of this theorem will comprise the remainder of the present section. To begin, we collect several observations and lemmas:

Lemma 5.6. *For k odd, let:*

$$\alpha_{k,s,m} := \sum_j [x_j, p_{j,k-2s+1} dx_m \omega^s] - [\omega^s x_m, a_{k-2s+1}].$$

We have $\alpha_{k,s,m} = 0 \pmod{L_4}$, for all k, s, m .

Proof. Let B denote the free algebra on generators z_0, z_1, \dots, z_{2n} , and define a surjective homomorphism $B \rightarrow A_{2n}$ by:

$$z_0 \mapsto \omega^s x_m, \quad z_i \mapsto x_i, \text{ for } i = 1, \dots, 2n.$$

We note that each expression $\alpha_{k,s,m}$ is alternating in the generators z_i , and lies in the image of $B_3(B)$. Lemma 5.1 of [AJ] therefore implies that $\alpha_{k,s,m} = 0 \pmod{L_4}$. \square

Lemma 5.7. *For any $a \in A_{2n}$ and k, l, s we have:*

$$l[a\omega^{s-k}, \omega^k] = k[a\omega^{s-l}, \omega^l] \pmod{L_4}.$$

Proof. We observe:

$$[a, \omega^k] = \sum_j [\omega^j a \omega^{k-1-j}, \omega] = k[a\omega^{k-1}, \omega] \pmod{L_4},$$

where we have used the containment $[M_3, A] \subset L_4$, from [BJ]. Replacing a by $\omega^{s-k}a$ gives:

$$l[\omega^{s-k}a, \omega^k] = kl[a\omega^{s-1}, \omega] = k[\omega^{s-l}a, \omega^l] \pmod{L_4}.$$

\square

Lemma 5.8. *We have $\psi_s(p_{l,k-2s+1}) = (2s+1)\phi_s(1 \otimes p_{l,k-2s+1})$*

Proof. Let $\tilde{p}_{l,k-2s+1}$ be a form such that $d\tilde{p}_{l,k-2s+1} = p_{l,k-2s+1}$. We compute:

$$\begin{aligned} \psi_s(p_{l,k-2s+1}) &= - \sum_i [\omega^s x_i, p_{l,k-2s+1} dx_{i+n}] - [\omega^s x_{i+n}, p_{l,k-2s+1} dx_i] \\ &= - \frac{1}{2} \sum_i [\omega^s x_i, [\tilde{p}_{l,k-2s+1}, x_{i+n}]] - [\omega^s x_{i+n}, [\tilde{p}_{l,k-2s+1}, x_i]] \\ &= - \sum_i -[x_i, p_{l,k-2s+1} dx_{i+n} \omega^s] + [x_{i+n}, p_{l,k-2s+1} dx_i \omega^s] \\ &\quad + 2[\tilde{p}_{l,k-2s+1}, \omega^{s+1}] \\ &= -\phi_s(1 \otimes p_{l,k-2s+1}) + 2[\tilde{p}_{l,k-2s+1}, \omega^{s+1}], \end{aligned}$$

by the Jacobi identity. By Lemma 5.7, we have:

$$[\tilde{p}_{l,k-2s+1}, \omega^{s+1}] = (s+1)[\tilde{p}_{l,k-2s+1} \omega^s, \omega].$$

For any $b \in A_{2n}$, the Jacobi identity implies:

$$[\omega, b] = \frac{1}{2} \sum_i [x_i, [x_{i+n}, b]] - [x_{i+n}, [x_i, b]],$$

so that

$$2[\tilde{p}_{l,k-2s+1}, \omega^{s+1}] = -2(s+1) \sum_i [x_i, [x_{i+n}, \tilde{p}_{l,k-2s+1} \omega^s]] - [x_{i+n}, [x_i, \tilde{p}_{l,k-2s+1} \omega^s]].$$

Thus, by converting to differential forms, and exchanging $p_{l,k-2s+1}$ and dx_i terms, we find:

$$\begin{aligned}\psi_s(p_{l,k-2s+1}) &= (2s+1) \sum_i [x_i, p_{l,k-2s+1} dx_{i+n} \omega^s] - [x_{i+n}, p_{l,k-2s+1} dx_i \omega^s] \\ &= (2s+1) \phi_s(1 \otimes p_{l,k-2s+1}),\end{aligned}$$

as desired. \square

Lemma 5.9. *For $k = n$, $\bar{y}_{k,s} \in L_3$*

Proof. Notice that the image of $y_{n,s}$ is 0 under the Fedosov product, thus

$$y_{n,s} \in L_2 \cap M_3 = L_3$$

by the key-lemma of [FS]. \square

Proposition 5.10. *We have the following:*

- (1) *The vectors $\bar{x}_{k,s}, \bar{y}_{k,s}, \bar{z}_{k,s}$ are \mathfrak{sp}_{2n} -highest weight vectors of weight $\rho_{k-2s}, \rho_{k+1-2s}, \rho_{k-1-2s}$, and total degree $k, k+1$, and $k+1$, respectively.*
- (2) *The vectors $\bar{x}_{k,s}, \bar{y}_{k,s}, \bar{z}_{k,s}$ satisfy the same equations as $\underline{x}_{k-2s}, \underline{y}_{k-2s}, \underline{z}_{k-2}$ in Proposition 2.10:*

$$\partial_l \bar{x}_{k,s} = 0, \text{ for } 1 \leq l \leq 2n,$$

$$\partial_l \bar{y}_{k,s} = \begin{cases} -X_{l,k+1}^T \bar{x}_{k,s}, & 1 \leq l \leq k, \\ \bar{x}_{k,s}, & l = k+1, \\ 0, & k+2 \leq l \leq 2n. \end{cases}$$

$$\partial_l \bar{z}_{k,s} = \begin{cases} -\sum_{j \geq k} Y_{lj}^T X_{kj}^T \bar{x}_{k,s}, & 1 \leq l \leq k-1, \\ -(n-k+2) Y_{k,l}^T \bar{x}_{k,s}, & k \leq l \leq n, \\ 0, & 1 \leq l-n \leq k-1, \\ (n-k+2) X_{k,l-n}^T \bar{x}_{k,s}, & k \leq l-n \leq n. \end{cases}$$

- (3) *The vectors $\bar{x}_{k,s}, \bar{y}_{k,s}, \bar{z}_{k,s}$ are the unique (up to scalars) vectors in $B_3(A_2)$ satisfying (1) and (2).*

Proof. Claim (1) follows from the fact that ϕ_s is a homomorphism of \mathfrak{sp}_{2n} -modules, together with the observation that all arguments of ϕ_s are clearly highest weight of the asserted weights. Claim (2) for $\bar{x}_{k,s}$ follows from the same claim for a_{k-2s} , which is clear. For (2), we compute for $k \neq n-1$ (the claim is clear for $k = n-1$):

For $1 \leq l \leq k - 2s + 1$:

$$\begin{aligned}
\partial_l \bar{y}_{k,s} &= \frac{k - 2s + 1}{k - 4s} \left(\sum_j \phi_s(\partial_l(x_j \otimes p_{j,k-2s+1}) + [x_j, p_{j,k-2s+1} dx_{l+n} \omega^s]) \right. \\
&\quad \left. - \frac{\psi_s(p_{l,k-2s+1})}{k - 2s + 1} - [\omega^s x_{l+n}, a_{k-2s+1}] \right) \\
&= \frac{k - 2s + 1}{k - 4s} \left(\phi_s(1 \otimes p_{l,k-2s+1}) - \frac{\psi_s(p_{l,k-2s+1})}{k - 2s + 1} + \alpha_{k,s,l+n} \right) \\
&= \phi_s(1 \otimes p_{l,k-2s+1}),
\end{aligned}$$

by Lemmas 5.6 and 5.8. For $k - 2s + 2 \leq l \leq 2n$:

$$\partial_l \bar{y}_{k,s} = \begin{cases} \frac{k-2s+1}{k-4s} \alpha_{s,k,l+n} = 0, & k - 2s + 2 \leq l \leq n \\ -\frac{k-2s+1}{k-4s} \alpha_{s,k,l-n} = 0, & n + 1 \leq l \leq 2n \end{cases},$$

by Lemma 5.6. For $\bar{z}_{k,s}$ we first consider the case $k - 2s = 1$; we have:

$$\begin{aligned}
\partial_l \bar{z}_{k,s} &= \sum_i [x_i, dx_{i+n} \wedge dx_{l+n} \omega^s] - [x_{i+n}, dx_i \wedge dx_{l+n} \omega^s] - [x_{l+n}, \omega^{s+1}] \\
&= \frac{1}{2} \left(\sum_i [x_i, [x_{i+n}, x_{l+n} \omega^s]] - [x_{i+n}, [x_i, x_{l+n} \omega^s]] \right) - [x_{l+n}, \omega^{s+1}] \\
&= [\omega, x_{l+n} \omega^s] + [\omega^{s+1}, x_{l+n}] = (s + 2)[\omega, x_{l+n} \omega^s] = (s + 2)\phi_s(1 \otimes a_1),
\end{aligned}$$

by the Jacobi identity.

For $k - 2s > 1$ we have, by direct computation:

$$\partial_l \bar{z}_{k,s} = \begin{cases} \phi_s(1 \otimes p_{l,k-2s-1} \omega - (n - (k - 2s) + 2)1 \otimes a_{k-2s-1} dx_{l+n}), & 1 \leq l \leq k - 2s - 1 \\ \phi_s((n - (k - 2s) + 2)1 \otimes a_{k-2s-1} dx_{l+n}), & k - 2s \leq l \leq n, \\ 0, & 1 \leq l - n \leq k - 2s - 1 \\ \phi_s(-(n - (k - 2s) + 2)1 \otimes a_{k-2s-1} dx_{l-n}), & k - 2s \leq l - n \leq n \end{cases}$$

The identities in Claim (2) involving $\bar{y}_{k,s}$ and $\bar{z}_{k,s}$ can now be read off directly, as in Proposition 2.10, recalling that ϕ_s is a morphism of \mathfrak{sp}_{2n} -modules.

Finally, for (3), we begin by noting that, inside $\mathcal{F}_{(2,1^{k-2s})}$ for k odd, the space of \mathfrak{sp}_{2n} -highest-weight vectors of weight ρ_m which are killed by all ∂_i 's is zero-dimensional if m is even, and one-dimensional if m is odd. This follows from the fact that the common kernel of all ∂_i 's is the generating \mathfrak{gl}_{2n} -module $(2, 1^k)$ \square

We now show:

Proposition 5.11. *The vector $\bar{x}_{k,s}$ is non-zero in $B_3(A_{2n})$.*

Proof. To show $\bar{x}_{k,s}$ is nonzero, we compute its image under θ as in Lemma 5.2. As before, we cannot use differential forms, so we use

$$a_{k-2s}^* = 2^{-(k-2s-1)/2} x_1 [x_2, x_3 [\dots [x_{k-2s-1}, x_{k-2s}]]].$$

Notice that d of the image of a_{k-2s}^* is a_{k-2s} . Recall that $\theta(x_1) = ez_0 + fz_1$, $\phi(x_i) = z_i$ for $i \geq 2$.

We compute:

$$\theta(a_{k-2s}) = (ez_0 + fz_1)z_2 \dots z_{k-2s}$$

Thus we find:

$$\theta(\bar{x}_{k,s}) = [e, f]z_0 \dots z_{k-2s}z_{1+n} \left(\sum_{i \geq 2} z_i z_{i+n} \right)^s$$

so $\bar{x}_{k,s}$ is non-zero in $B_3(B)$, and thus in $B_3(A_{2n})$. \square

Corollary 5.12. *The vectors $\bar{y}_{k,s}$, $\bar{z}_{k,s}$ are non-zero in $B_3(A_{2n})$.*

Proof. We have shown that $\bar{x}_{k,s}$ is non-zero in $B_3(A_{2n})$, and it lies in the orbit of both $\bar{y}_{k,s}$ and $\bar{z}_{k,s}$. \square

By Lemma 3.1, $B_3(A'_{2n})$ is a quotient of $B_3(A_{2n})$ by the subspace $L_3 \cap \langle \omega \rangle$. Clearly $\bar{y}_{k,s}$, and thus $\bar{x}_{k,s}$, is divisible by ω for all $s \geq 0$, while $v_{k,s}$ and $\bar{z}_{k,s}$ are divisible by ω for all $s \geq 1$. We thus have a surjection

$$\pi : \bigoplus_{\substack{k=1, \\ k \text{ odd}}}^{n-1} (\mathcal{F}_k / Y_k \oplus \mathcal{F}_{(2,1^k)}) \twoheadrightarrow B_3(A'_{2n}).$$

We have computed directly in MAGMA that $v_{k,0}$ and $z_{k,0}$ are non-zero in $B_3(A'_{2n})$ when $2n = 4, 6$. Based on this, we conjecture that π is an isomorphism for all n :

Conjecture 5.13. *The vectors $v_{k,0}$ and $z_{k,0}$ are non-zero in $B_3(A'_{2n})$, and thus the H_{2n} -module composition factors for $B_3(A'_{2n})$ are:*

$$\mathcal{F}_{(2,1^k)}, \mathcal{F}_{(1^k)} / T_k, Z_k / X_k \quad \text{for } k \text{ odd, } 1 \leq k \leq n-1.$$

REFERENCES

- [A.J] Noah Arbesfeld, David Jordan, *New results on the lower central series quotients of a free associative algebra*, Journal of Algebra, Volume 323, Issue 6 (2010): 1813-1825. arXiv:0902.4899.
- [BB] Martina Balagovic, Anirudha Balasubramanian, *On the Lower Central Series Quotients of a Graded Associative Algebra*, Journal of Algebra, Volume 328, Issue 1, 15 February 2011, Pages 287-300. arXiv:1004.3735v1
- [B.J] Asilata Bapat, David Jordan, *Lower central series of free algebras in symmetric tensor categories*. arXiv:1001.1375v2.

- [DE] Galyna Dobrovolska, Pavel Etingof. *An upper bound for the lower central series quotients of a free associative algebra*. International Mathematics Research Notices, Vol. 2008, rnn039. arXiv:0801.1997.
- [FH] William Fulton, Joe Harris, *Representation Theory: A First Course* Springer, 1991.
- [FS] Boris Feigin, Boris Shoikhet, *On $[A, A]/[A, [A, A]]$ and on a W_n -action on the consecutive commutators of free associative algebras*. Math. Res. Lett. 14 (2007), no. 5, 781–795.
- [G] Victor Guillemin, *Hodge Theory*. Course notes, Spring 1997. <http://www-math.mit.edu/~vwg/shlomo-notes.pdf>
- [R] A.N Rudakov, *Irreducible Representations of Infinite-Dimensional Lie Algebras of Types S and H* 1975 Math. USSR Izv. 9 465 doi:10.1070/IM1975v009n03ABEH001487.
- [BCP] Wieb Bosma, John Cannon, Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., 24 (1997), 235265.

DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE, MA 02139, DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN TX 78712